GENERAL INFORMATION ABOUT ITERATIVE SOLVERS

There are methods that can be used to solve a set of linear equations that are based on iteration. An initial estimate of the parameters is estimated and then the equations are solved, yielding an updated version of the parameters. These new values are then inserted back into the equations and process continues until the desired solution is reached. The two common methods which are discussed here are the Jacobi method and the Gauss-Seidel method.

1. Jacobi Method

Given a set of linear equations:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 \ldots + a_{nn}x_n &= b_n
\end{aligned}
$$

Solve for $x_1, x_2, \ldots, x_n$

We begin by rearranging these equations in the form of solving for the unknown parameters one equation at a time:

$$
\begin{aligned}
x_1 &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}}x_2^{(0)} - \frac{a_{13}}{a_{11}}x_3^{(0)} - \cdots - \frac{a_{1n}}{a_{11}}x_n^{(0)} \\
x_2 &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}}x_1^{(0)} - \frac{a_{23}}{a_{22}}x_3^{(0)} - \cdots - \frac{a_{2n}}{a_{22}}x_n^{(0)} \\
&\vdots \\
x_n &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}}x_1^{(0)} - \frac{a_{n2}}{a_{nn}}x_2^{(0)} - \cdots - \frac{a_{n,n-1}}{a_{nn}}x_{n-1}^{(0)}
\end{aligned}
$$

The superscript (0) indicates the initial estimate of the parameters. For the first iteration, these parameters are given the value zero. The equations are then solved which results in an updated value of the parameters. These current estimates are then inserted back into the equations and a newer set of parameters is arrived at by solving these equations. The process continues until the solution converges.

Example:

$$
\begin{cases}
7x_1 + 3x_2 + x_3 = 18 \\
2x_1 - 9x_2 + 4x_3 = 12 \\
x_1 - 4x_2 + 12x_3 = 6
\end{cases}
$$

Rearrange these equations:

$$x_1 = \frac{18}{7} - \frac{3}{7}x_2 - \frac{1}{7}x_3 \approx 2.571 - 0.429x_2 - 0.143x_3$$
$$x_2 = -\frac{12}{9} + \frac{2}{9}x_1 + \frac{4}{9}x_3 \approx -1.333 + 0.222x_1 + 0.444x_3$$
$$x_3 = \frac{6}{12} - \frac{1}{12}x_1 + \frac{4}{12}x_2 \approx 0.5000 - 0.083x_1 + 0.333x_2$$

Use initial estimates: $x_1^{(0)} = x_2^{(0)} = x_3^{(0)} = 0$

$$x_1^{(1)} = 2.571 - 0.429x_2^{(0)} - 0.143x_3^{(0)} = 2.571 - 0.429(0) - 0.143(0) = 2.571$$
$$x_2^{(1)} = -1.333 + 0.222x_1^{(0)} + 0.444x_3^{(0)} = -1.333 + 0.222(0) + 0.444(0) = -1.333$$
$$x_3^{(1)} = 0.5000 - 0.083x_1^{(0)} + 0.333x_2^{(0)} = 0.5000 - 0.083(0) + 0.333(0) = 0.5000$$

Insert these updated estimates back into original equation again, yielding:

$$x_1^{(2)} = 2.571 - 0.429x_2^{(1)} - 0.143x_3^{(1)} = 2.571 - 0.429(-1.333) - 0.143(0.5000) = 3.071$$
$$x_2^{(2)} = -1.333 + 0.222x_1^{(1)} + 0.444x_3^{(1)} = -1.333 + 0.222(2.571) + 0.444(0.5000) = -0.540$$
$$x_3^{(2)} = 0.5000 - 0.083x_1^{(1)} + 0.333x_2^{(1)} = 0.5000 - 0.083(2.571) + 0.333(-1.333) = -0.159$$

Continue this process until the desired results are obtained.

The table below shows the solutions arrived at after each iteration:

Iteration	X(1)	X(2)	X(3)
1	2.55143	-1.33333	.5000
2	3.07143	-.53968	-.15873
3	2.82540	-.72134	.06415
4	2.87141	-.67695	.2410
5	2.85811	-.68453	.03506
6	2.85979	-.68261	.03365

The exact solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2.859 \\ -0.683 \\ 0.034 \end{bmatrix}$, as we can see, the solutions using the Jacobi iterative method are very close.

$$x_1^{(1)} = 2.571 - 0.429x_2^{(0)} - 0.143x_3^{(0)} = 2.571 - 0.429(0) - 0.143(0) = 2.571$$
$$x_2^{(1)} = -1.333 + 0.222x_1^{(1)} + 0.444x_3^{(0)} = -1.333 + 0.222(2.571) + 0.444(0) = -0.762$$
$$x_3^{(1)} = 0.5000 - 0.083x_1^{(1)} + 0.333x_2^{(1)} = 0.5000 - 0.083(2.571) + 0.333(-0.762) = 0.033$$

Next iteration:

$$x_1^{(2)} = 2.571 - 0.429x_2^{(1)} - 0.143x_3^{(1)} = 2.571 - 0.429(-0.762) - 0.143(0.033) = 2.893$$
$$x_2^{(2)} = -1.333 + 0.222x_1^{(2)} + 0.444x_3^{(1)} = -1.333 + 0.222(2.893) + 0.444(0.033) = -0.676$$
$$x_3^{(2)} = 0.5000 - 0.083x_1^{(2)} + 0.333x_2^{(2)} = 0.5000 - 0.083(2.893) + 0.333(-0.676) = 0.034$$

The table below shows the solutions arrived at after each iteration:

Iteration	X(1)	X(2)	X(3)
1	2.55143	-.76190	.03175
2	2.89342	-.67624	.03347
3	2.85646	-.6869	.03406
4	2.85957	-.68273	.03412

2. Gauss-Seidel Iterative Method

The Gauss-Seidel iterative method of solving for a set of linear equations can be thought of as just an extension of the Jacobi method. Start out using an initial value of zero for each of the parameters. Then, solve for x_1 as in the Jacobi method. When solving for x_2, insert the just computed value for x_1, and so on. In other words, for each calculation, the most current estimate of the parameter value is used.

Let's use the Gauss-Seidel iterative method to solve the last example: