QUESTION 22: BASICS OF FDM METHOD (FINITE DIFFERENCES, NUMERICAL DIFFERENTIATION).

Finite differences are numerical methods to approximate solutions of differential equations using finite difference to approximate derivatives.Let be $y = f(x)$, from the definition of derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

thererfore an approximation is

$$f'(x) \approx \frac{f(x+h) - f(x)}{h} \tag{1}$$

and (1) is the **forward difference** equation for the first derivative. Let be $y_i = f(x)$, $y_{i+1} = f(x+h)$, equation (1) can be written as

$$\frac{dy}{dx} = \frac{y_{i+1} - y_i}{h} \ .$$

For the first derivative we also have the **backward difference** approximation,

$$\frac{dy}{dx} = \frac{y_i - y_{i-1}}{h}$$

and the **central difference** approximation,

$$\frac{dy}{dx} = \frac{y_{i+1} - y_{i-1}}{2h} \ .$$

For the **second derivative** the **central difference** is given by

$$\frac{d^2y}{dx^2} = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \ .$$

EXAMPLE: PREDADOR-PREY MODEL

Find a numerical solution of the system of ordinary differential equations (*Lotka-Volterra equations*):

$$\begin{aligned} \frac{dx}{dt} &= ax - bxy \\ \frac{dy}{dt} &= -cy + dxy \end{aligned} \tag{2}$$

with initial conditions $x(0) = x_0$ and $y(0) = y_0$.

Where x and y are the number of prey and predators, respectively, a the prey growth rate, c the predator death rate, and b and d the rate characterizing the effect of the predator-prey interaction on prey death and predator growth, respectively.

NUMERICAL SOLUTION

By using forward difference we have

$$\frac{dx}{dt} = \frac{x_{t+1} - x_t}{\Delta t} \text{ and } \frac{dy}{dt} = \frac{y_{t+1} - y_t}{\Delta t} \tag{3}$$

By replacing (3)into (2) we get

$$\begin{aligned} \frac{x_{t+1} - x_t}{\Delta t} &= ax_t - bx_t y_t \\ \frac{y_{t+1} - y_t}{\Delta t} &= -cy_t + dx_t y_t \end{aligned}$$

and solving for x_{t+1} and y_{t+1} we get the explicit scheme

$$\begin{aligned} x_{t+1} &= (1 + a\Delta t)\, x_t - bx_t y_t \Delta t \\ y_{t+1} &= (1 - c\Delta t)\, y_t + dx_t y_t \Delta t \end{aligned} \tag{4}$$

Algorithm 1 predator_prey_model.m solve numerically the system of differential equations (2)

```
% ----------------------------------------------------------
% predator_prey_model.m
% Solve system of differential equation
% x' = ax - bxy
% y'=  -cy + dxy
% ----------------------------------------------------------
%parameters in the model
a = 1.2;
b = 0.6;
c = 0.8;
d = 0.3;
% initial conditions
x = 2;
y = 1;
T = 20; %final time
dt = 0.01; %time step
% Initialize vectors solution
X = [x]; %predator
Y = [y]; %prey
a1 = 1+a*dt;
c1 = 1-c*dt;
for i = 1:dt:T
% Solution using forward difference
x = a1*x - b*x*y*dt;   % equation for prey a1 = 1+a*dt
y = c1*y + d*x*y*dt;   % equation for predator c1 = 1-c*dt
X = [X;x]; %prey in each time
Y = [Y;y]; %predator in each time
end
t = 0:dt:T;
t = t(1:size(X));
% plot solutions
plot(t,X,'b'),grid on, hold on
plot(t,Y,'r')
legend('prey','predator')
xlabel('time','FontSize', 12)
ylabel('population','FontSize',12)
title('Predador - Prey model', 'FontSize',14)
```

By implementing the scheme (4) in Matlab and using the parameter given in Table 1, and initial conditions $x_0 = 2$ and $y_0 = 1$ for $t \in [0, 20]$ we get the plot given in figure 1. The code predator_prey_model.m is given in Algorithm 1.

Parameter	value	Interpretation
a	1.2	prey growth rate
b	0.6	prey death because of predator
c	0.8	predator death rate
d	0.3	predator growth because of prey

Table 1: Parameter used to solve numerically the predator-prey mode

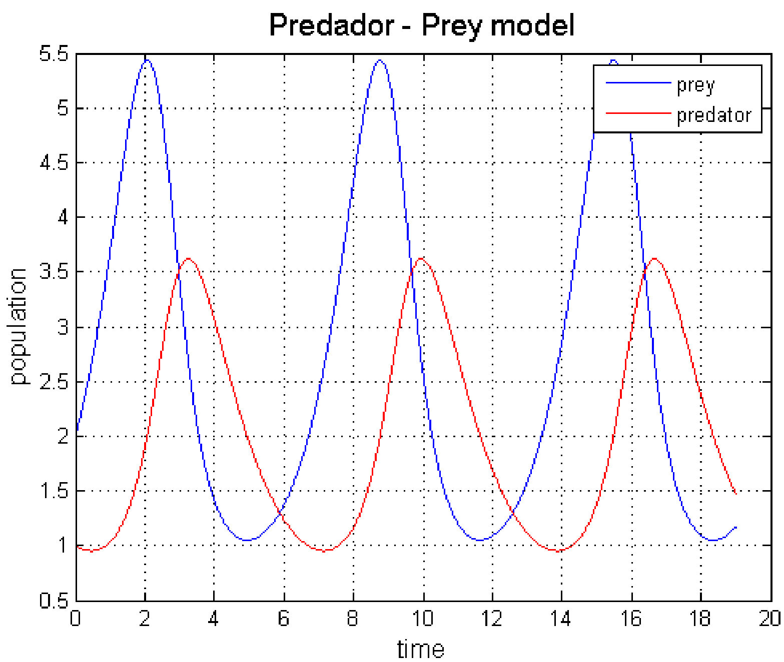

Figure 1: Numerical solution for the predator-prey model.