GAUSSIAN ELIMINATION & LU DECOMPOSITION

1. Gaussian Elimination

It is easiest to illustrate this method with an example. Let’s consider the system of equations
\[
\begin{align*}
    x - 3y + z &= 4 \\
    2x - 8y + 8z &= -2 \\
    -6x + 3y - 15z &= 9
\end{align*}
\]

To solve for \(x\), \(y\), and \(z\), we must eliminate some of the unknowns from some of the equations. Consider adding -2 times the first equation to the second equation and also adding 6 times the first equation to the third equation:
\[
\begin{align*}
    x - 3y + z &= 4 \\
    0x + 2y + 6z &= -10 \\
    0x + 15y - 9z &= 33
\end{align*}
\]

We have now eliminated the \(x\) term form the last two equations, Now simplify the last two equations by 2 and 3, respectively:
\[
\begin{align*}
    x - 3y + z &= 4 \\
    0x + y + 3z &= -5 \\
    0x + 5y - 3z &= 11
\end{align*}
\]

To eliminate the \(y\) term in the last equation, multiply the second equation by -5 and add it to the third equation:
\[
\begin{align*}
    x - 3y + z &= 4 \\
    0x + y + 3z &= -5 \\
    0x + 0y - 18z &= 36
\end{align*}
\]

From the third equation, we can get \(z = -2\), substituting this into the second equation yields \(y = -1\). Using both of these results in the first equation gives \(x = 3\). This process of progressively solving for the unknowns is back-substitution.

Now, let’s see this example in matrix:

First, convert the system of equations into an augmented matrix:
\[
\begin{bmatrix}
    1 & -3 & 1 & 4 \\
    2 & -8 & 8 & -2 \\
    -6 & 3 & -15 & 9
\end{bmatrix}
\]

\[
\begin{bmatrix}
    1 & -3 & 1 & 4 \\
    0 & -2 & 6 & -10 \\
    0 & -15 & -9 & 33
\end{bmatrix}
\]

\[
\begin{bmatrix}
    1 & -3 & 1 & 4 \\
    0 & -1 & 3 & -5 \\
    0 & -5 & -3 & 11
\end{bmatrix}
\]

\[
\begin{bmatrix}
    1 & -3 & 1 & 4 \\
    0 & -1 & 3 & -5 \\
    0 & 0 & -18 & 36
\end{bmatrix}
\]
2. LU Composition

If \( A \) is a square matrix and it can be factored as \( A = LU \) where \( L \) is a lower triangular matrix and \( U \) is an upper triangular matrix, then we say that \( A \) has an **LU-Decomposition** of \( LU \).

If \( A \) is a square matrix and it can be reduced to a row-echelon form, \( U \), without interchanging any rows, then \( A \) can be factored as \( A = LU \) where \( L \) is a lower triangular matrix.

LU decomposition of a matrix is not unique.

There are three factorization methods:

- Crout Method: \( \text{diag} (U) = 1; u_{ii} = 1 \)
- Doolittle Method: \( \text{diag} (L) = 1; l_{ii} = 1 \)
- Choleski Method: \( \text{diag} (U) = \text{diag} (L) ; u_{ii} = l_{ii} \)

To solve several linear systems \( Ax = b \) with the same \( A \), and \( A \) is big, we would like to avoid repeating the steps of Gaussian elimination on \( A \) for every different \( B \). The most efficient and accurate way is LU-decomposition, which in effect records the steps of Gaussian elimination. This is Doolittle Method.

Without pivoting:

\[
Ax = b
\]

\[
LUx = b
\]

To solve this, first we solve \( Ly = b \) for \( y \) by forward-substitution method,
then solve \( Ux = y \) for \( x \) by backward-substitution method.

With pivoting:

\[
Ax = b
\]

\[
PAx = Pb, \text{ where } P \text{ is permutation matrix.}
\]

\[
LUx = Pb
\]

To solve this, first we solve \( Ly = Pb \) for \( y \) by forward-substitution method,
then solve \( Ux = y \) for \( x \) by backward-substitution method.
The main idea of the LU decomposition is to record the steps used in Gaussian elimination on A in the places where the zero is produced.

Let’s see an example of LU-Decomposition without pivoting:

\[
A = \begin{bmatrix}
1 & -2 & 3 \\
2 & -5 & 12 \\
0 & 2 & -10
\end{bmatrix}
\]

The first step of Gaussian elimination is to subtract 2 times the first row form the second row. In order to record what was done, the multiplier, 2, into the place it was used to make a zero.

\[
\begin{bmatrix}
1 & -2 & 3 \\
\text{(2)} & -1 & 6 \\
0 & 2 & -10
\end{bmatrix}
\]

There is already a zero in the lower left corner, so we don’t need to eliminate anything there. We record this fact with a (0). To eliminate \(a_{32}\), we need to subtract -2 times the second row from the third row. Recording the -2:

\[
\begin{bmatrix}
1 & -2 & 3 \\
\text{(2)} & -1 & 6 \\
\text{(0)} & \text{(-2)} & 2
\end{bmatrix}
\]

Let U be the upper triangular matrix produced, and let L be the lower triangular matrix with the records and ones on the diagonal:

\[
L = \begin{bmatrix}
1 & 0 & 0 \\
2 & 1 & 0 \\
0 & -2 & 1
\end{bmatrix}, \quad U = \begin{bmatrix}
1 & -2 & 3 \\
0 & -1 & 6 \\
0 & 0 & -10
\end{bmatrix}
\]

Then,

\[
LU = \begin{bmatrix}
1 & 0 & 0 \\
2 & 1 & 0 \\
0 & -2 & 1
\end{bmatrix} \begin{bmatrix}
1 & -2 & 3 \\
0 & -1 & 6 \\
0 & 0 & -10
\end{bmatrix} = \begin{bmatrix}
1 & -2 & 3 \\
2 & -5 & 12 \\
0 & 2 & -10
\end{bmatrix} = A
\]

Go back to the first example, rewrite the system of equation into matrix equation:

\[
\begin{bmatrix}
1 & -3 & 1 \\
2 & -8 & 8 \\
-6 & 3 & -15
\end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 9 \end{bmatrix}
\]

\[Ax = b\]

\[A: \begin{bmatrix}
1 & -3 & 1 \\
2 & -8 & 8 \\
-6 & 3 & -15
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & -3 & 1 \\
\text{(2)} & -2 & 6 \\
\text{(-6)} & -15 & -9
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & -3 & 1 \\
\text{(2)} & -2 & 6 \\
\text{(-6)} & \left(-\frac{15}{2}\right) & -54
\end{bmatrix}\]
\[ A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -6 & -\frac{15}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 1 \\ 0 & -2 & 6 \\ 0 & 0 & -54 \end{bmatrix} = LU \]

**LU Decomposition**

\[ \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -6 & -\frac{15}{2} & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 9 \end{bmatrix} \]

**Solve Ly = b using forward substitution**

\[ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -10 \\ 108 \end{bmatrix} \]

\[ \begin{bmatrix} 1 & -3 & 1 \\ 0 & -2 & 6 \\ 0 & 0 & -54 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -10 \\ 108 \end{bmatrix} \]

**Solve Ux = y using backward substitution**

\[ \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix} \]